

Benchmarking Economic Efficiency: Technical and Allocative Fundamentals

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Presentation and Outline

- **The measurement and decomposition of the economic efficiency of organizations** (firms, branches, departments, etc.), is receiving increasing attention from theoretical and applied scholars interested in identifying the technical and allocative causes underlying suboptimal market behavior, i.e., why organizations fail to achieve an economic goal.
- In this talk we will review the ‘**state-of-the-art**’ in measuring econ. efficiency:
 - 1) **Definitions** of profit, profitability, cost, and revenue efficiencies.
 - 2) **Duality theory** enables their consistent decomposition into technical and allocative efficiency.
 - 3) **Decomposing** economic efficiency: Theory (DEA) and Practice (BEE.jl)
 - 4) **Extensions**: endogenous models and dynamic models

Definitions

What is Economic (In)Efficiency?

- Economic efficiency **defines** as the ability of firms to achieve optimal performance in terms of a reference economic benchmark (**objective function ≡ optimizing behavior**): profit, profitability, cost, or revenue.
- For an individual firm, **economic efficiency analysis compares its observed profit, profitability, cost or revenue, with the optimal economic benchmark** within the industry or corporation.

$$\Pi I(x, y, w, p) = \Pi(w, p) - (p \cdot y - w \cdot x) \geq 0 \quad | \quad CE(x, y, w) = \frac{C(y, w)}{w \cdot x} \leq 1$$

- It represents a systematic way of **evaluating performance against those of the competitors** and therefore contribute with relevant quantity and price indicators complementing popular measures like ROA, SWOT,....

Historical Remarks

- Economic efficiency analysis **was introduced by Farrell (1957)**, inspired by **Debreu's (1951)** “coefficient of resource utilization”. Therefore, it dates back to the very first paper on firm's efficiency. Farrell introduced the canonical model decomposing economic efficiency (cost) into technical and allocative efficiency: $CE = TE \times AE$, where AE is a residual.
- Shephard (1953, 1970) mathematically formalized the concept of technical efficiency in terms of distance functions (input) and developed the duality with the supporting economic function (cost), which is the corner stone in economic efficiency analysis.

Historical Remarks

- Shephard **fell short** from formalizing economic efficiency *a la* Farrell. Färe and Primont (1995) **present this formalization and connect both authors** (cost and revenue).
- Moreover, they resort to Kooopmans (1951) ‘Activity Analysis’ and its associated Data Envelopment Analysis (DEA) optimizing techniques (introduced by Charnes et al., 1978) to render economic efficiency operational.
- They built upon Färe et al. (1985), who presented a comprehensive analysis of economic efficiency (cost and revenue, profit with the hyperbolic distance function), but did not resort to duality theory.

Duality Theory

Duality at work !

- Cost efficiency (multiplicative):

- Observed cost:

$$C_o = \mathbf{w} \cdot \mathbf{x}_o = \sum_{m=1}^M w_m x_{om}$$

- Cost function:

$$C(\mathbf{y}_o, \mathbf{w}) = \min_x \left\{ \mathbf{w} \cdot \mathbf{x} : \mathbf{x} \in L(\mathbf{y}_o) \right\}, \quad \mathbf{w} \in \mathbb{R}_{++}^M, \mathbf{y}_o \geq 0_N$$

- Technical efficiency:

$$TE_{R(I)}(\mathbf{x}_o, \mathbf{y}_o) = \min_{\theta} \left\{ \theta \leq 1 : (\theta \mathbf{x}_o) \in L(\mathbf{y}_o) \right\}$$

- Duality theory relates the cost function and technology:

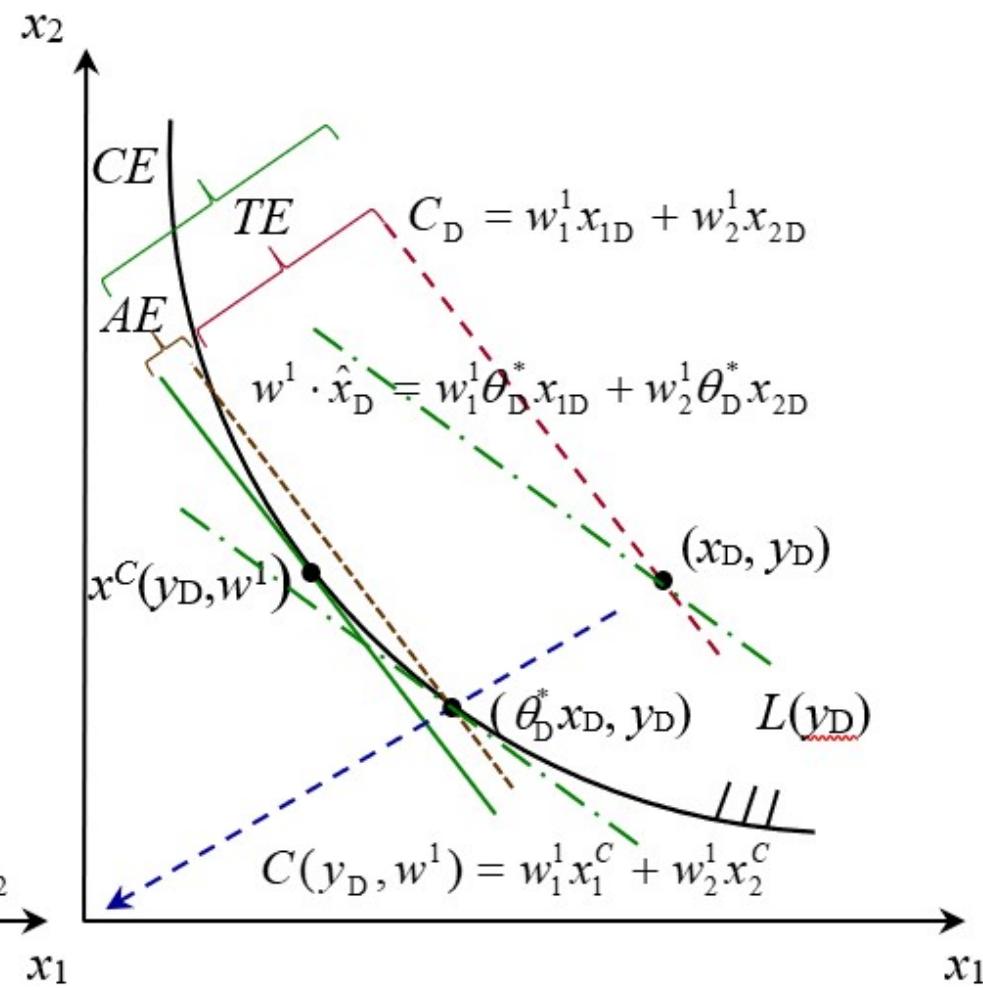
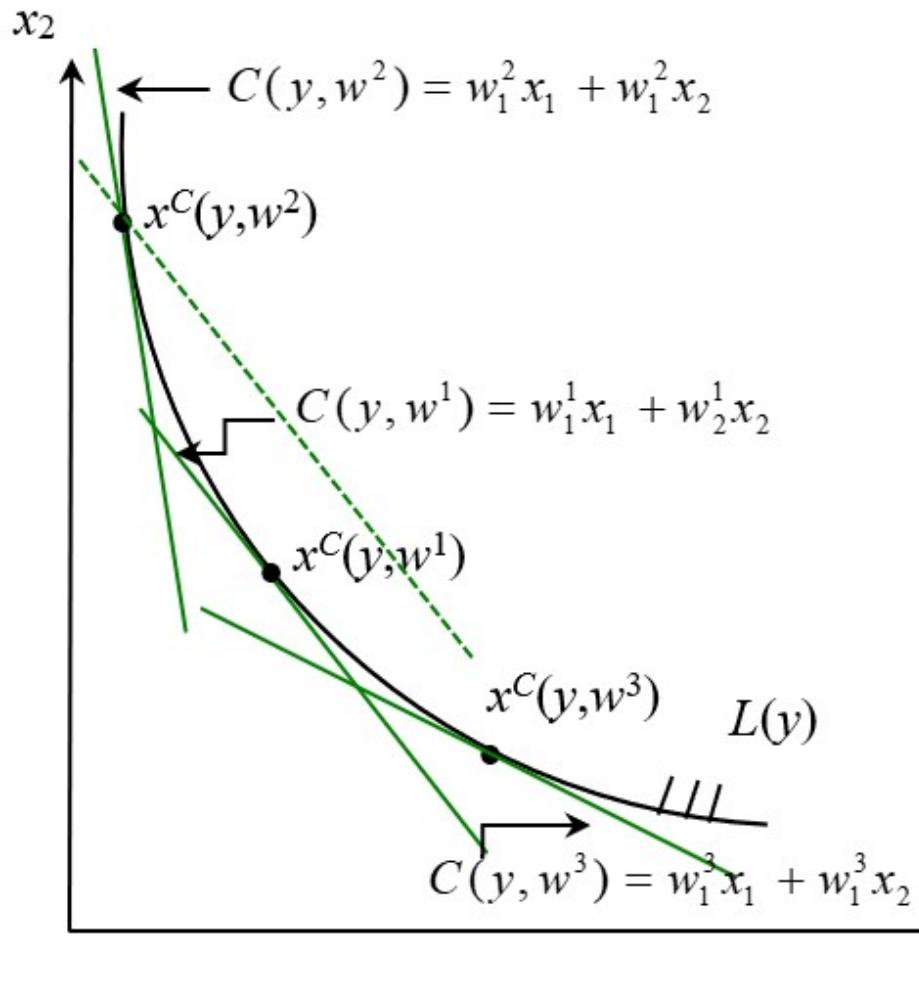
$$L(\mathbf{y}_o) = \left\{ \mathbf{x} : \mathbf{w} \cdot \hat{\mathbf{x}} = \mathbf{w} \cdot \theta \mathbf{x} \geq C(\mathbf{y}_o, \mathbf{w}), \theta \leq 1, \forall \mathbf{w} \in \mathbb{R}_{++}^M, \mathbf{y}_o \geq 0_N \right\}$$

- Embedding the so-called Fenchel-Mahler inequality:

$$C(\mathbf{y}_o, \mathbf{w}) \leq \mathbf{w} \cdot \theta \mathbf{x}_o = \mathbf{w} \cdot \hat{\mathbf{x}}_o$$

Duality at work !

- Duality between the cost function and input TE (shadow and market prices).



Duality at work !

- And we can **define the following cost efficiency measure** relating the cost function, observed cost and the technical efficiency measure:

$$CE_{R(I)}(x_o, y_o, w) = \frac{C(y_o, w)}{w \cdot x_o} \leq \theta$$

- Closing the inequality, we obtain the **decomposition of cost efficiency**:

$$CE_{R(I)}(x_o, y_o, w) = TE_{R(I)}(x_o, y_o) \times AE_{R(I)}(x_o, y_o, w)$$

$$\frac{C(y_o, w)}{\sum_{m=1}^M w_m x_{om}} = \underbrace{\theta^*}_{\text{Technical Efficiency}} \times \underbrace{AE_{R(I)}(x_o, y_o, w)}_{\text{Allocative Efficiency}} \leq 1$$

Cost Efficiency

Historical Remarks

- Economic efficiency analysis remained confined to the cost and revenue dimensions until Chambers et al. (1996, 1998) introduced **the duality between the profit function and the directional distance function, DDF** (Luenberger, 1992) \Rightarrow Additive decomposition: $\Pi I = TI_{EM(G)} + AI_{EM(G)}$, where AI is a residual. Notation.
- Zofío and Prieto (2006) formalized the **dual relationship between the profitability function and the generalized distance function, GDF** (Chavas and Cox, 1999) \Rightarrow Multiplicative decomposition: $\Gamma E = TE_{GDF} \times AE_{GDF}$, where AE is a residual. Notation.
- The cost and revenue analyses represent particular cases (nested models) of the profit and profitability decompositions. Notation.

Duality: Additive Economic Efficiency

Profit maximization (VRS)

$$N\pi_{DDF(G)} = \frac{\Pi(p, w) - (p \cdot y_o - w \cdot x_o)}{(p \cdot g_y + w \cdot g_x)} \geq D_T(x_o, y_o; g_x, g_y)$$

Normalization factor, NF_{DDF}

$$\frac{\Pi(p, w) - (p \cdot y_o - w \cdot x_o)}{N\pi_{DDF(G)}(x_o, y_o, w, p)} = \underbrace{D_T(x, y; g_x, g_y)}_{\Pi_{DDF(G)}(x_o, y_o)} + AI_{DDF(G)}(x_o, y_o, \tilde{w}, \tilde{p})$$

Cost minimization (VRS)

Revenue maximization (VRS)

$$NCI = \frac{w \cdot x_o - C(y, w)}{w \cdot g_x} \geq D_T(x_o, y_o; g_x, 0)$$



$$\frac{w \cdot x_o - C(y, w)}{w \cdot g_x} = \underbrace{D_T(x, y; g_x, 0)}_{\Pi_{DDF(I)}(x_o, y_o; g_x, 0)} + AI_{DDF(I)}(x_o, y_o, \tilde{w})$$

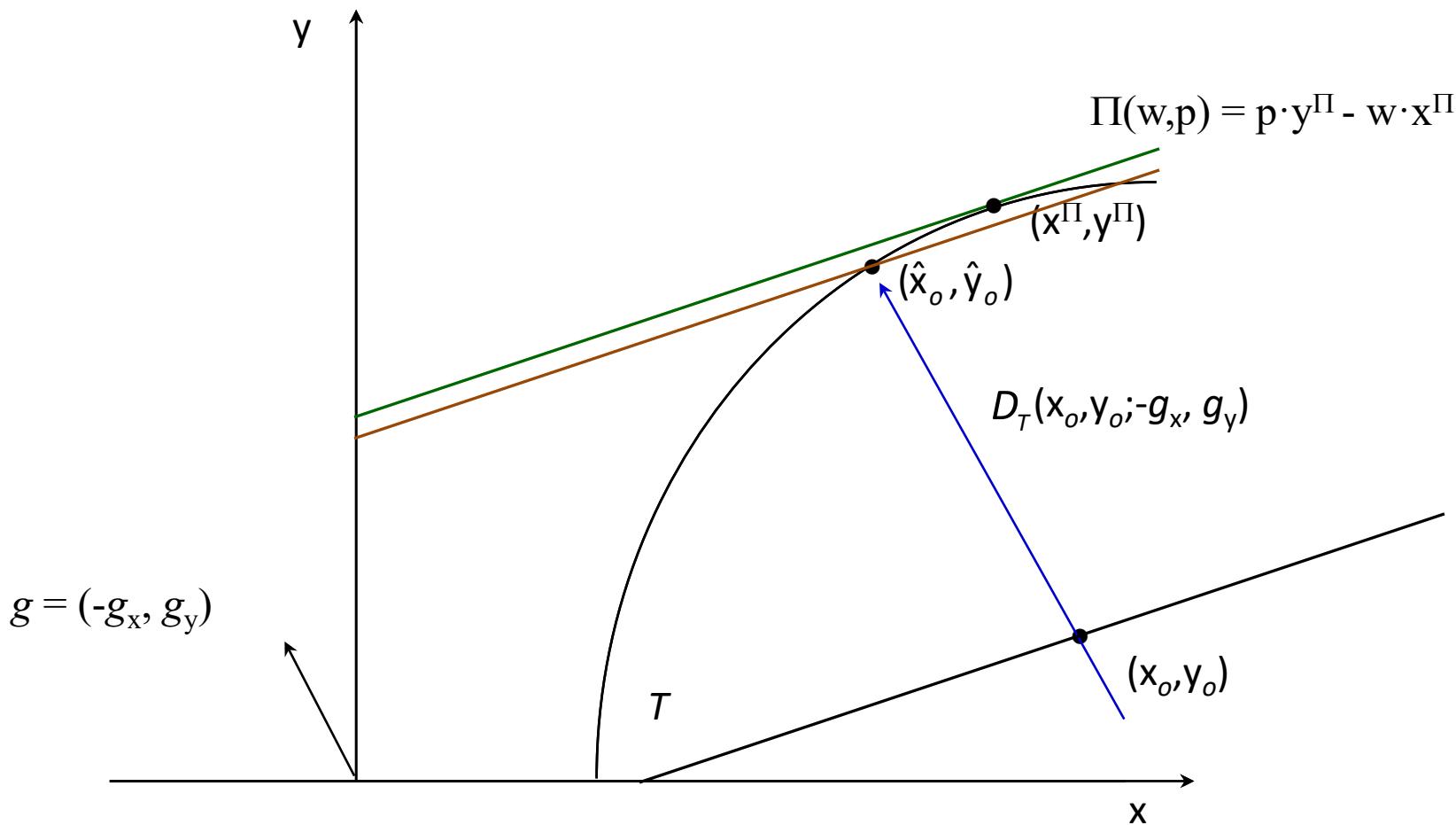
$$NRI = \frac{R(x_o, p) - p \cdot y_o}{p \cdot g_y} \geq D_T(x_o, y_o; 0, g_y)$$



$$\frac{R(x_o, p) - p \cdot y_o}{NRI(x_o, y_o, p)} = \underbrace{D_T(x, y; 0, g_y)}_{\Pi_{DDF(O)}(x_o, y_o; 0, g_y)} + AI_{DDF(O)}(x_o, y_o, \tilde{p})$$

Duality: Profit and Profitability Efficiency

- Profit efficiency decomposition



Duality: Multiplicative Economic Efficiency

Profitability maximization (CRS)

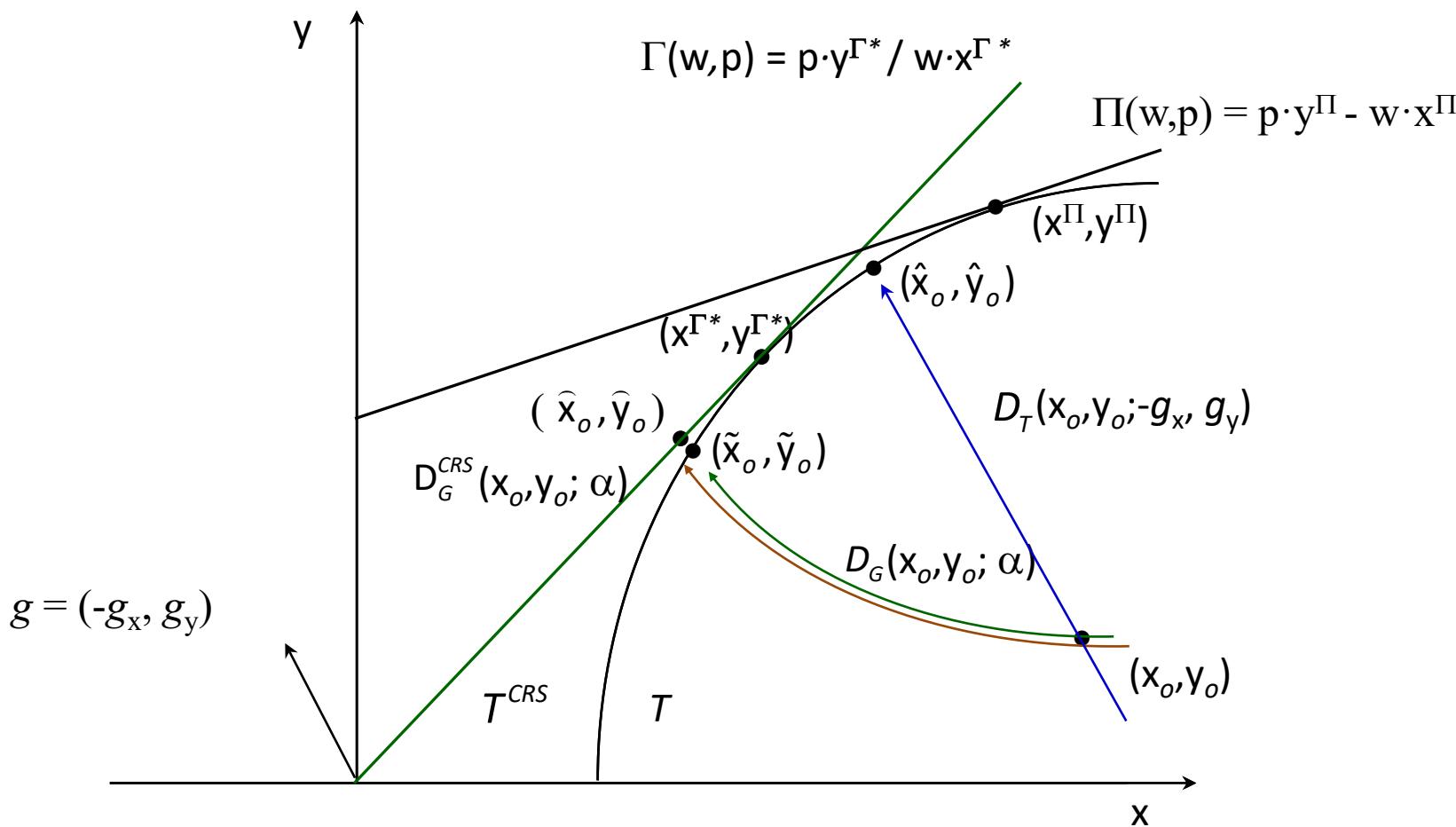
$$\begin{aligned}
 \Gamma E(x_o, y_o, w, p) &= \frac{p \cdot y_o / w \cdot x_o}{\Gamma(p, w)} \leq D_G^{CRS}(x, y; \alpha) \\
 &\downarrow \\
 \underbrace{\frac{p \cdot y_o / w \cdot x_o}{\Gamma(p, w)}}_{\Gamma E(x_o, y_o, w, p)} &\leq \underbrace{\frac{D_G^{CRS}(x, y; \alpha)}{TE_G(x_o, y_o)}}_{\frac{D_G(x, y; \alpha)}{SE_G(x_o, y_o)}} \\
 &\downarrow \\
 \underbrace{\frac{p \cdot y_o / w \cdot x_o}{\Gamma(p, w)}}_{\Gamma E(x_o, y_o, w, p)} &= \underbrace{\frac{D_G(x, y; \alpha)}{TE_G(x_o, y_o)}}_{\frac{D_G(x, y; \alpha)}{SE_G(x_o, y_o)}} \times AE_G(x_o, y_o, w, p; \alpha) \\
 &\searrow \quad \swarrow \\
 &\text{Cost minimization (VRS)} \quad \text{Revenue maximization (VRS)}
 \end{aligned}$$

$$CE(x_o, x_o, w) = \frac{C(y, w)}{w \cdot x_o} \leq D_G(x_o, y_o; 0) \quad RE(x_o, x_o, p) = \frac{p \cdot y_o}{R(x, p)} \leq D_G(x_o, y_o; 1)$$

$$\begin{aligned}
 \underbrace{\frac{C(y_o, w)}{w \cdot x_o}}_{CE(x_o, x_o, w)} &= \underbrace{\frac{D_G(x_o, y_o; 0)}{TE_{G(I)}(x_o, x_o)}}_{AE_{G(I)}(x_o, y_o, w; 0)} \quad \underbrace{\frac{p \cdot y_o}{R(x_o, p)}}_{RE(x_o, x_o, p)} = \underbrace{\frac{D_G(x_o, y_o; 1)}{TE_{G(O)}(x_o, x_o)}}_{AE_{G(O)}(x_o, y_o, p; 1)}
 \end{aligned}$$

Duality: Profit and Profitability Efficiency

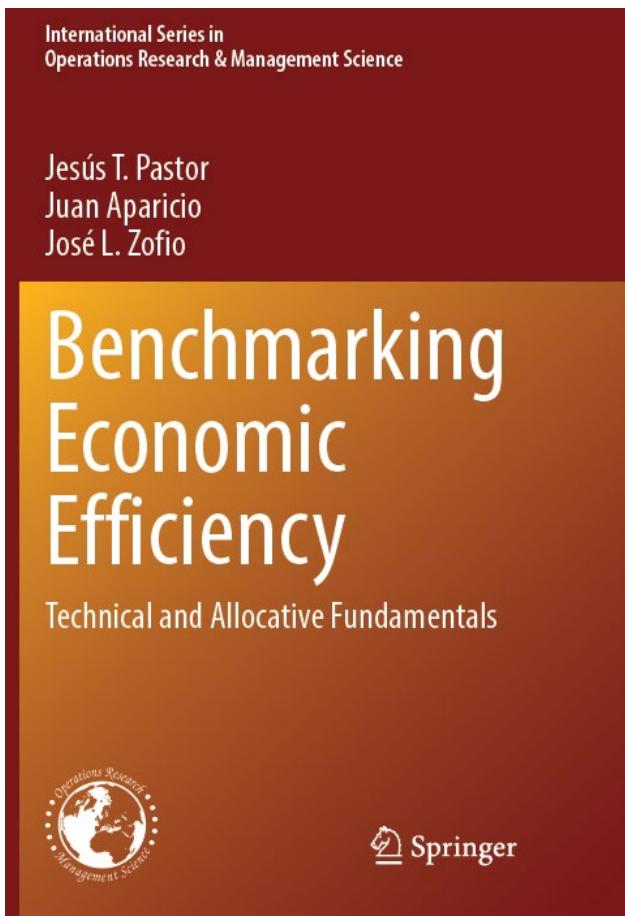
- Profit and profitability efficiency decomposition



Measuring and Decomposing Economic Efficiency

References: Models of Economic Efficiency

Theory



Practice

BenchmarkingEconomicEfficiency.jl:
Economic efficiency measurement with
Data Envelopment Analysis

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Abstract

BenchmarkingEconomicEfficiency.jl is a new package for Julia that includes functions for the measurement and decomposition of the economic efficiency of organizations. It aims at a wide range of scholars and professionals working in the fields of economics, engineering, management science and operations research. Using mathematical programming methods known as Data Envelopment Analysis, the package includes code to decompose profit, profitability, cost and revenue efficiency, defined as ratios or differences. Depending on the functional form of economic efficiency it can be decomposed, either multiplicative or additively, into two separate and mutually exclusive components representing technical (production) efficiency and allocative (market) efficiency. The former can be measured through a number of DEA efficiency models. The package implements traditional decompositions based on the radial (input or output) technical efficiency measures, and new ones based on the generalized distance function, the directional distance function, DDF (including novel extensions like the modified DDF, reverse DDF, or generalizations based on Hölder norms), the Russell measures, the additive measures like the weighted additive distance function or slack based measure, etc. This paper describes the methodology and implementation of the functions, and reports numerical results using a common dataset on financial institutions.

Keywords: data envelopment analysis, economic efficiency decomposition, technical efficiency, allocative efficiency, Julia.

Software

- Julia or GitHub repositories

The screenshot shows a GitHub repository page for the package `BenchmarkingEconomicEfficiency.jl`. The repository is public and was forked from `javierbarbero/BenchmarkingEconomicEfficiency.jl`. The main branch is `main`, which is 4 commits ahead of the upstream branch. The repository has 2 branches and 0 tags. The README.md file contains the following content:

```
BenchmarkingEconomicEfficiency.jl

A Julia package for economic efficiency measurement using Data Envelopment Analysis (DEA).

Documentation Build Status Coverage
docs stable docs dev CI passing codecov 100%
```

The package provides functions for economic efficiency measurement using Data Envelopment Analysis (DEA). It is an extension of the `DataEnvelopmentAnalysis.jl` package. The package is developed for Julia 1.0 and above on Linux, macOS, and Windows. It uses the JuMP modelling language for mathematical optimization with solvers GLPK and Ipopt.

Installation

The package can be installed with the Julia package manager:

```
julia> using Pkg; Pkg.add("BenchmarkingEconomicEfficiency")
```

The GitHub page also includes sections for `About`, `Languages` (showing 100% Julia), and a GitHub cat icon.

www.benchmarkingeconomicefficiency.com

Benchmarking Economic Efficiency

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Benchmarking Economic Efficiency

BenchmarkingEconomicEfficiency.jl is a new package for **Julia** that includes functions for the measurement and decomposition of the economic efficiency of organizations. Using *Data Envelopment Analysis* methods, the package includes code to decompose **profit, profitability, cost and revenue efficiency**, defined as ratios or differences. Depending on the functional form of economic efficiency it can be decomposed, either multiplicative or additively, into two separate and mutually exclusive components representing technical efficiency and allocative efficiency. The former can be measured through a number of DEA efficiency models. The package implements traditional decompositions based on the radial (input or output) technical efficiency measures, and new ones based on the generalized distance function, the directional distance function, DDF (including novel extensions like the modified DDF, reverse DDF, or generalizations based on Hölder norms), the Russell measures, the additive measures like the weighted additive distance function or slack based measure, etc.

Documentation is available [here](#).

Illustration: The canonical model for the decomposition of normalized profit inefficiency into technical and allocative terms:

$$\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^N p_n y_{on} - \sum_{m=1}^M w_m x_{om} \right)}{NF_{EM(G)}} =$$

Norm. Profit Inefficiency

$$= \underbrace{TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o)}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o, \hat{\mathbf{w}}, \hat{\mathbf{p}})}_{\text{Norm. Allocative Inefficiency}} \geq 0.$$

Roadmap: Novel Models of Economic Efficiency

- Multiplicative decomp. of economic efficiency: $EE = TE \times AE$ (Weak EM)
 - ✓ Farrell (Radial Efficiency) Measures (Shephard's Distance Functions):
 - ✓ Cost Efficiency & Revenue Efficiency
 - ✓ Generalized Distance Function (Hyperbolic Efficiency):
 - ✓ Profitability Efficiency: Profitability = Revenue / Cost = R / C
- Additive decomp. of economic *inefficiency*: $EI = TI + AI$ (Strong IM)
 - ✓ Russell Inefficiency Measure (Russell)
 - ✓ Weighted Additive Distance Function (WADF)
 - ✓ Enhanced Russell Inefficiency Measure (SBM)
 - ✓ Directional Distance Function (Weak IM) (DDF)
 - ✓ Hölder Distance Function (Weak IM) (HDF)
 - ✓ Modified Distance Function (Weak IM) (MDF)
 - ✓ Reverse Distance Function (RDF)
 - ✓ General Direct Approach (GDA)

Cost Inefficiency
Revenue Inefficiency
Profit Inefficiency (Profit $\Pi=R-C$)

Additive Models: Profit Inefficiency (Gen. Model)

- Normalized Profit Inefficiency: usual technological axioms (VRS)

- Observed profit: $\Pi_o = (\mathbf{p} \cdot \mathbf{y}_o - \mathbf{w} \cdot \mathbf{x}_o)$

- Maximum profit: $\Pi(\mathbf{w}, \mathbf{p}) = \max_{\mathbf{x}, \mathbf{y}} \{ \mathbf{p} \cdot \mathbf{y} - \mathbf{w} \cdot \mathbf{x} \mid \mathbf{x} \geq X\lambda, \mathbf{y} \leq Y\lambda, \mathbf{e}\lambda = 1, \lambda \geq 0 \}$

- Technical inefficiency: $TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o) \geq 0 \Rightarrow$ Duality

- Fenchel-Mahler Inequality: $\Pi(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p}) / NF_{EM(G)} \geq TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o)$

$$\begin{aligned} N\Pi(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}) &= \underbrace{\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^N p_n y_{on} - \sum_{m=1}^M w_m x_{om} \right)}{NF_{EM(G)}}}_{\text{Norm. Profit Inefficiency}} = \\ &= \underbrace{TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o)}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}})}_{\text{Norm. Allocative Inefficiency}} \geq 0. \end{aligned}$$

The Russell Inefficiency Measure (1/2)

- Halická and Trnovská (2018) show how to solve Russell IM as a semidefinite programming (SDP) model and develop economic duality results.

Technical Inefficiency: $TI = 1 - TE$

$$TE_{RM(G)}(\mathbf{x}_o, \mathbf{y}_o) = \min_{\boldsymbol{\theta}, \boldsymbol{\phi}, \boldsymbol{\lambda}} \frac{1}{M+N} \left(\sum_{m=1}^M \theta_m + \sum_{n=1}^N \frac{1}{\phi_n} \right)$$

s.t.

$$\begin{aligned} \sum_{j=1}^J \lambda_j x_{jm} &= \theta_m x_{om}, & m &= 1, \dots, M \\ \sum_{j=1}^J \lambda_j y_{jn} &= \phi_n y_{on}, & n &= 1, \dots, N \\ \sum_{j=1}^J \lambda_j &= 1, \\ \theta_m &\leq 1, & m &= 1, \dots, M \\ \phi_n &\geq 1, & n &= 1, \dots, N \\ \lambda_j &\geq 0, & j &= 1, \dots, J \end{aligned}$$

Profit Inefficiency: $NIII$

$$\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^N p_n y_{on} - \sum_{m=1}^M w_m x_{om} \right)}{(M+N) \underbrace{\min \{w_1 x_{o1}, \dots, w_M x_{oM}, p_1 y_{o1}, \dots, p_N y_{oN}\}}_{\text{Norm. Profit Inefficiency}}} =$$

$$\underbrace{\left[1 - \frac{1}{M+N} \left(\sum_{m=1}^M \theta_m^* + \sum_{n=1}^N \frac{1}{\phi_n^*} \right) \right]}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{RM(G)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}})}_{\text{Norm. Allocative Inefficiency}} \geq 0.$$

The Russell Inefficiency Measure (2/2)

- BenchmarkingEconomicEfficiency.jl resorts to the JuMP.jl package by Dunning *et al.* (2017), along with the ‘Ipopt’ solver, Wächter and Biegler (2006).

```
julia> deaprofitrussell(X, Y, W, P, names = banks)
```

(Session 2B CEYP)

```
Russell Profit DEA Model  
DMUs = 31; Inputs = 3; Outputs = 2  
Orientation = Graph; Returns to Scale = VRS
```

| | Profit | Technical | Allocative |
|---------------------------|------------|------------|------------|
| ----- | | | |
| Export-Import Bank | 3.03911e-5 | 4.81347e-6 | 2.55776e-5 |
| Bank of Taiwan | 0.179835 | 2.71764e-9 | 0.179835 |
| Taipei Fubon Bank | 0.220019 | 2.34592e-7 | 0.220019 |
| Bank of Kaohsiung | 4.95721 | 0.319545 | 4.63766 |
| Land Bank | 1.3703e-7 | 5.04573e-8 | 8.65722e-8 |
| ... | | | |
| Hwatai Bank | 136.01 | 0.54445 | 135.466 |
| Cota Bank | 43.6129 | 0.528602 | 43.0843 |
| Industrial Bank of Taiwan | 1.56449 | 1.7079e-5 | 1.56447 |
| Bank SinoPac | 0.422128 | 0.125224 | 0.296904 |
| Shin Kong Bank | 9.67215 | 0.435376 | 9.23677 |
| ----- | | | |

The Weighted Additive D.F. (1/2)

- Aparicio *et al.* (2016) measures graph technical inefficiency based solely on input and output slacks and develop economic duality results.

Technical Inefficiency: TI

$$TI_{WADF(G)}(\mathbf{x}_o, \mathbf{y}_o, \rho^-, \rho^+) = \max_{\mathbf{s}^-, \mathbf{s}^+, \boldsymbol{\lambda}} \sum_{m=1}^M \rho_m^- s_m^- + \sum_{n=1}^N \rho_n^+ s_n^+$$

s.t.

$$\begin{aligned} & \sum_{j=1}^J \lambda_j x_{jm} + s_m^- \leq x_{om}, \quad m = 1, \dots, M \\ & - \sum_{j=1}^J \lambda_j y_{jr} + s_n^+ \leq -y_{on}, \quad n = 1, \dots, N \\ & \sum_{j=1}^J \lambda_j = 1, \\ & s_m^- \geq 0, \quad m = 1, \dots, M \\ & s_n^+ \geq 0, \quad n = 1, \dots, N \\ & \lambda_j \geq 0, \quad j = 1, \dots, J \end{aligned}$$

Profit Inefficiency: NII

$$\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^N p_n y_{on} - \sum_{m=1}^M w_m x_{om} \right)}{\min \underbrace{\left\{ \frac{w_1}{\rho_1^-}, \dots, \frac{w_M}{\rho_M^-}, \frac{p_1}{\rho_1^+}, \dots, \frac{p_N}{\rho_N^+} \right\}}_{\text{Norm. Profit Inefficiency}}} =$$

$$= \underbrace{\sum_{m=1}^M \rho_m^- s_m^{-*} + \sum_{n=1}^N \rho_n^+ s_n^{+*}}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{WADF(G)}(\mathbf{x}_o, \mathbf{y}_o, \rho^-, \rho^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}})}_{\text{Norm. Allocative Inefficiency}} \geq 0$$

```
julia> deaprofitadd(X, Y, W, P, :MIP, names = banks)
```

The Weighted Additive D.F. (2/2)

● Weight options:

- :**Ones**, indicates that the vectors or weights corresponding to inputs and outputs are set to one, $(\rho^-, \rho^+) = (\mathbf{1}, \mathbf{1})$, and therefore the unweighted additive model introduced by Charnes *et al.* (1985) is calculated;
- :**MIP**. Calculates the Measure of Inefficiency Proportions (MIP) (see Cooper, Park, and Pastor (1999)) considering $(\rho^-, \rho^+) = (1/\mathbf{x}_o, 1/\mathbf{y}_o)$, where $1/\mathbf{x}_o = (1/x_{1o}, \dots, 1/x_{Mo})$ and $1/\mathbf{y}_o = (1/y_{1o}, \dots, 1/y_{No})$;
- :**RAM**. Calculates the Range Adjusted Measure of Inefficiency (RAM) (see Cooper *et al.* (1999)) considering $(\rho^-, \rho^+) = (1/(M+N)\mathbf{R}^-, 1/(M+N)\mathbf{R}^+)$, where $\mathbf{R}^- = (R_1^-, \dots, R_M^-)$ with $R_m^- = \max_{1 \leq j \leq J} \{x_{jm}\} - \min_{1 \leq j \leq J} \{x_{jm}\}$, $m = 1, \dots, M$, and $\mathbf{R}^+ = (R_1^+, \dots, R_N^+)$ with $R_n^+ = \max_{1 \leq j \leq J} \{y_{jn}\} - \min_{1 \leq j \leq J} \{y_{jn}\}$;
- :**BAM**. Calculates the Bounded Adjusted Measure of inefficiency (BAM) (see Cooper, Pastor, Borras, Aparicio, and Pastor (2011)) considering $\rho^- = 1/\left[(M+N)(\underline{x}_o - \bar{x})\right]$, where $\underline{\mathbf{x}} = \left(\underline{x}_1, \dots, \underline{x}_M\right)$ with $\underline{x}_m = \min_{1 \leq j \leq J} \{x_{jm}\}$, $m = 1, \dots, M$, and $\rho^+ = 1/[(M+N)(\bar{y} - y_o)]$, where $\bar{\mathbf{y}} = (\bar{y}_1, \dots, \bar{y}_N)$ with $\bar{y}_n = \max_{1 \leq j \leq J} \{y_{jn}\}$, $n = 1, \dots, N$; and, finally,
- :**Normalized**. Calculates the normalized weighted additive model (see Lovell and Pastor (1995)) considering $(\rho^-, \rho^+) = (1/\sigma^-, 1/\sigma^+)$, where $\sigma^- = (\sigma_1^-, \dots, \sigma_M^-)$ is the vector of sample standard deviations of inputs and $\sigma^+ = (\sigma_1^+, \dots, \sigma_N^+)$ is the vector of sample standard deviations of outputs.

The Enhanced Russell Graph IM (or SBM)

- Introduced by Pastor *et al.* (1999) and Tone (2001). Aparicio *et al.* (2017) develop the economic duality results.

Technical Inefficiency: TI

$$TE_{ERG=SBM(G)}(\mathbf{x}_o, \mathbf{y}_o) = \min_{\mathbf{s}^-, \mathbf{s}^+, \boldsymbol{\lambda}} \frac{1 - \frac{1}{M} \sum_{m=1}^M \frac{s_m^-}{x_{om}}}{1 + \frac{1}{N} \sum_{n=1}^N \frac{s_n^+}{y_{on}}}$$

s.t.

$$\sum_{j=1}^J \lambda_j x_{jm} = x_{om} - s_m^-, \quad m = 1, \dots, M$$

$$\sum_{j=1}^J \lambda_j y_{jn} = y_{on} + s_n^+, \quad n = 1, \dots, N$$

$$\sum_{j=1}^J \lambda_j = 1,$$

$$s_m^- \geq 0, \quad m = 1, \dots, M$$

$$s_n^+ \geq 0, \quad n = 1, \dots, N$$

$$\lambda_j \geq 0, \quad j = 1, \dots, J$$

Profit Inefficiency: $NIII$

$$\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^N p_n y_{on} - \sum_{m=1}^M w_m x_{om} \right)}{\delta_{(\mathbf{x}_o, \mathbf{y}_o, \mathbf{p}, \mathbf{w})} \left(1 + \frac{1}{N} \sum_{n=1}^N \frac{s_n^{+*}}{y_{on}} \right)} =$$

Norm. Profit Inefficiency

$$= \underbrace{\left(\frac{\frac{1}{N} \sum_{n=1}^N \frac{s_n^{+*}}{y_{on}} + \frac{1}{M} \sum_{m=1}^M \frac{s_m^{-*}}{x_{om}}}{\left(1 + \frac{1}{N} \sum_{n=1}^N \frac{s_n^{+*}}{y_{on}} \right)} \right)}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{ERG=SBM(G)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{p}}, \tilde{\mathbf{w}})}_{\text{Norm. Allocative Inefficiency}} \geq 0$$

```
julia> deaprofiterg(X, Y, W, P, names = banks)
```

The Directional Distance Function (1/2)

- Chambers *et al.* (1996) proposed the DDF based on Luenberger's shortage function, and Chambers *et al.* (1998) developed the economic duality results.

Technical Inefficiency: TI

$$TI_{DDF(G)}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_x^-, \mathbf{g}_y^+) = \max_{\beta, \lambda} \beta$$

s.t.

$$\sum_{j \in J} \lambda_j x_{jm} \leq x_{om} - \beta g_{x_m}^-, \quad m = 1, \dots, M$$
$$\sum_{j \in J} \lambda_j y_{jn} \geq y_{on} + \beta g_{y_n}^+, \quad n = 1, \dots, N$$
$$\sum_{j \in J} \lambda_j = 1$$
$$\lambda_j \geq 0, \quad j \in J.$$

Profit Inefficiency: NII

$$\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^N p_n y_{on} - \sum_{m=1}^M w_m x_{om} \right)}{\underbrace{\sum_{m=1}^M w_m g_{om}^- + \sum_{n=1}^N p_n y_{on}^+}_{\text{Norm. Profit Inefficiency}}} =$$
$$= \underbrace{\beta_{DDF(G)}^*}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{DDF(G)}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_x^-, \mathbf{g}_y^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}})}_{\text{Norm. Allocative Inefficiency}} \geq 0$$

```
julia> deaprofit(X, Y, W, P, Gx = :Mean, Gy = :Mean, names = banks)
```

The Directional Distance Function (2/2)

- Options:
 - :Observed. $Gx = :Observed$, $Gy = :Observed$. Sets the input and output directional vector to the observed quantities of the firm under evaluation: $\mathbf{g} = (\mathbf{g}_x^-, \mathbf{g}_y^+) = (\mathbf{x}_o, \mathbf{y}_o)$;
 - :Ones. $Gx = :Ones$, $Gy = :Ones$. Sets the directional vectors to one: $\mathbf{g} = (\mathbf{g}_x^-, \mathbf{g}_y^+) = (1, 1)$;
 - :Mean. $Gx = :Mean$, $Gy = :Mean$. Sets the directional vector to $\mathbf{g} = (\bar{x}_M, \bar{y}_N)$, where each common m-th and n-th element corresponds to the arithmetic mean of each input and output: $\bar{x}_m = \sum_{j=1}^J x_{mj}/J$ and $\bar{y}_n = \sum_{j=1}^J y_{nj}/J$, respectively;
 - :Zeros. Either $Gx = :Zeros$ or $Gy = :Zeros$. Sets the output or input direction to zero. Combined with the previous options allows calculating input oriented DDFs, e.g., $\mathbf{g} = (\mathbf{x}_o, \mathbf{0}_N)$, $\mathbf{g} = (\mathbf{1}_M, \mathbf{0}_N)$ or $\mathbf{g} = (\bar{\mathbf{x}}_M, \mathbf{0}_N)$, and output oriented DDFs, e.g., $\mathbf{g} = (\mathbf{0}_M, \mathbf{y}_o)$, $\mathbf{g} = (\mathbf{0}_M, \mathbf{1}_N)$, or $\mathbf{g} = (\mathbf{0}_M, \bar{\mathbf{y}}_N)$. These oriented DDFs are used in forthcoming sections to decompose cost and revenue inefficiency based on the input-and output-oriented DDFs, respectively. Finally,
 - :Monetary. $Gx = :Monetary$, $Gy = :Monetary$. Following (Zofío, Pastor, and Aparicio 2013, p. 213) this option returns profit inefficiency, including its technical and allocative components, with values expressed in monetary units (e.g., dollars). To

The Hölder Distance Function (1/2)

- Briec (1998) introduced the Hölder DF and its economic duality ($h = 2$)

Technical Inefficiency: TI ($h = 2$, Euclidean)

$$TI_{WHölder}(G)(\mathbf{x}_o, \mathbf{y}_o, 2) = \min_{\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}, \beta, \gamma} \sqrt{\sum_{m=1}^M (x_{om} - x_m)^2 + \sum_{n=1}^N (y_n - y_{on})^2}$$

s.t.

$$\begin{aligned} & \sum_{j=1}^J \lambda_j x_{jm} \leq x_m, \\ & \sum_{j=1}^J \lambda_j y_{jn} \geq y_n, \\ & \sum_{j=1}^J \lambda_j = 1, \\ & \beta = 0, \\ & \max_{\beta, \gamma} \beta \\ & \text{s.t. } \sum_{j=1}^J \gamma_j x_{jm} \leq x_m - \beta, \\ & \sum_{j=1}^J \gamma_j y_{jn} \geq y_n + \beta, \\ & \sum_{j=1}^J \gamma_j = 1, \\ & \lambda_j, \gamma_j \geq 0, \\ & x_m \geq 0, \\ & y_n \geq 0, \end{aligned}$$

$m = 1, \dots, M$

$n = 1, \dots, N$

$j = 1, \dots, J$

$m = 1, \dots, M$

$n = 1, \dots, N$

Profit Inefficiency: $NPII$

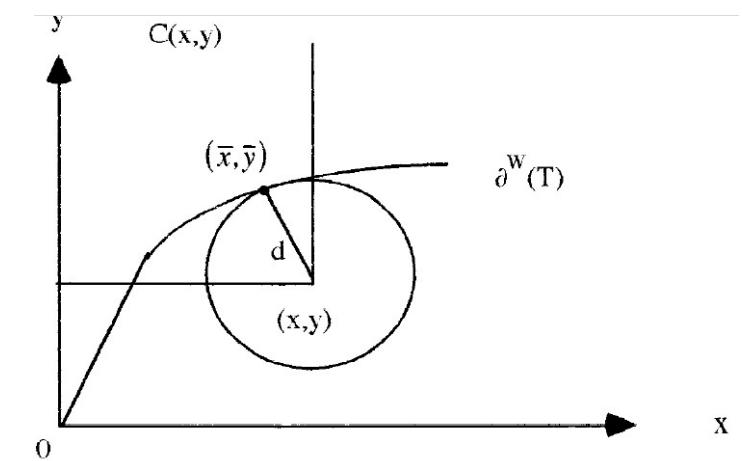
$$\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^N p_n y_{on} - \sum_{m=1}^M w_m x_{om} \right)}{\|(\mathbf{w}, \mathbf{p})\|_q} =$$

Norm. Profit Inefficiency

$$\sqrt{\sum_{m=1}^M (x_{om} - x_m^*)^2 + \sum_{n=1}^N (y_n^* - y_{on})^2} + AI_{WHölder}(G)(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}}, h) \geq 0$$

Graph Technical Inefficiency

Norm. Profit Inefficiency



The Hölder Distance Function (1/2)

- BenchmarkingEconomicEfficiency.jl implements the Hölder function under the following norms: unit ($h = 1$), infinitum ($h = \infty$), and $h = 2$, i.e., the Euclidean distance (SOS conditions).

```
julia> deaprofitholder(X, Y, W, P, l = Inf, weight = true, names = banks)
```

- Options:

| Norm | Unweighted * | Weighted: 'weight=true' |
|---------------|--|--|
| ℓ_1 | $\mathbf{g}_x = (0, \dots, 1_{(m')}, \dots, 0), m' = 1, \dots, M$ | $\mathbf{g}_x = (0, \dots, x_{(m'o)}, \dots, 0), m' = 1, \dots, M$ |
| | $\mathbf{g}_y = (0, \dots, 1_{(n')}, \dots, 0), n' = 1, \dots, N$ | $\mathbf{g}_y = (0, \dots, y_{(n'o)}, \dots, 0), n' = 1, \dots, N$ |
| ℓ_∞ | $\mathbf{g} = (\mathbf{g}_x^-, \mathbf{g}_y^+) = (\mathbf{1}_M, \mathbf{1}_N)$ | $\mathbf{g} = (\mathbf{g}_x^+, \mathbf{g}_y^+) = (\mathbf{x}_{oM}, \mathbf{y}_{oN})$ |
| ℓ_2 | $(\mathbf{1}_M, \mathbf{1}_N)$ | $\left((1/x_{oM})^2, (1/y_{oN})^2 \right)$ |

Notes: * Unweighted results are obtained by default when omitting `weight = true` from the syntax.

The Modified Directional Distance Function

- Aparicio et al. (2013) introduced the Modified Directional Distance Function and developed the economic duality.

Technical Inefficiency: TI

$$TI_{MDDF(G)}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_x^-, \mathbf{g}_y^+) = \max_{\beta_x, \beta_y, \lambda} \beta_x + \beta_y$$

s.t.

$$\sum_{j=1}^J \lambda_j x_{jm} \leq x_{om} - \beta_x g_{x_m}, \quad m = 1, \dots, M$$
$$\sum_{j=1}^J \lambda_j y_{jn} \geq y_{on} + \beta_y g_{y_n}, \quad n = 1, \dots, N$$
$$\sum_{j=1}^J \lambda_j = 1,$$
$$\lambda_j \geq 0, \quad j = 1, \dots, J$$
$$\beta_x, \beta_y \geq 0$$

Profit Inefficiency: $NPII$

$$\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^N p_n y_{on} - \sum_{m=1}^M w_m x_{om} \right)}{\underbrace{C_o}_{\text{Norm. Profit Inefficiency}}} =$$
$$= \underbrace{\beta_x^* + \beta_y^*}_{\text{Technical Inefficiency}} + \underbrace{AI_{DDF(G)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \mathbf{g}_x^-, \mathbf{g}_y^+, \tilde{\mathbf{p}})}_{\text{Norm. Allocative Inefficiency}} \geq 0$$

```
julia> deaprofitmddf(X, Y, W, P, Gx = :Observed, Gy = :Observed, names = banks)
```

The Reverse Directional Distance Function

- Pastor *et al.* (2016) introduced the Reverse Directional Distance Function and developed the economic duality.
- To calculate the *RDDF* for an efficiency measure, $EM(G)$, we determine the directional vector, $\mathbf{g} = (\mathbf{g}_x^-, \mathbf{g}_y^+)$, connecting the firm and its projection: $(\hat{\mathbf{x}}_o, \hat{\mathbf{y}}_o)$
- Therefore, the *RDDF* “translates” any inefficiency measure into a *DDF*. Solves the “weakly efficiency” problem of the original *DDF*.
- Profit Inefficiency: $N\Pi /$

$$\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^N p_n y_{on} - \sum_{m=1}^M w_m x_{om} \right)}{\underbrace{\sum_{m=1}^M w_m g_{om}^- + \sum_{n=1}^N p_n g_{on}^+}_{\text{Norm. Profit Inefficiency}}} =$$
$$= \underbrace{\beta_{RDDF(EM(G), F_J, \hat{F}_J)}^*}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{RDDF(EM(G), F_J, \hat{F}_J)}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{g}_x^-, \mathbf{g}_y^+, \tilde{\mathbf{w}}, \tilde{\mathbf{p}})}_{\text{Norm. Allocative Inefficiency}} \geq 0.$$

```
julia> deaprofitrddf(X, Y, W, P, :ERG, names = banks)
```

The General Direct Approach (1/2)

- Pastor *et al.* (2022) introduced the General Direct Approach. It does not rely on duality theory. Again, we require a projection ($EM(G)$) on the frontier.
- Profit Inefficiency decomposition based on slacks (weights = prices)

$$\begin{aligned}\Pi I(x_o, y_o, w, p) &= \Pi(w, p) - \Pi_o = \Pi(w, p) - (p \cdot y_o - w \cdot x_o) = \\ &= (p \cdot s_{oEM(G)}^{+*} + w \cdot s_{oEM(G)}^{-*}) + (\Pi(w, p) - (p \cdot \hat{y}_{oEM(G)} - w \cdot \hat{x}_{oEM(G)})) = \\ &= \underbrace{(p \cdot s_{oEM(G)}^{+*} + w \cdot s_{oEM(G)}^{-*})}_{\text{Profit Technological Gap}} + \underbrace{\Pi(\hat{x}_{oEM(G)}, \hat{y}_{oEM(G)}, w, p)}_{\text{Profit Allocative Inefficiency}}.\end{aligned}$$

- Normalizing by the $EM(G)$

$$\begin{aligned}\Pi I(x_o, y_o, w, p) &= \\ TI_{EM(G)}(x_o, y_o) \times \left(\frac{p \cdot s_{oEM}^{+*} + w \cdot s_{oEM}^{-*}}{TI_{EM(G)}(x_o, y_o)} \right) + \Pi I(\hat{x}_{oEM(G)}, \hat{y}_{oEM(G)}, w, p)\end{aligned}$$

The General Direct Approach (2/2)

- Normalizing factor

$$NF_{EM(G)}^{GDA}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p}) = \begin{cases} \frac{(\mathbf{p} \cdot \mathbf{s}_{EM(G)}^{+*} + \mathbf{w} \cdot \mathbf{s}_{EM}^{-*})}{TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o)}, & (\mathbf{x}_o, \mathbf{y}_o) \neq (\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}) \\ k_o \$, \quad k_o > 0 , & (\mathbf{x}_o, \mathbf{y}_o) = (\hat{\mathbf{x}}_{oEM(G)}, \hat{\mathbf{y}}_{oEM(G)}) \end{cases}$$

- Profit Inefficiency:

$$\underbrace{\frac{\Pi(\mathbf{w}, \mathbf{p}) - \left(\sum_{n=1}^N p_n y_{on} - \sum_{m=1}^M w_m x_{om} \right)}{NF_{EM(G)}^{GDA}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p})}}_{\text{Norm. Profit Inefficiency}} =$$
$$= \underbrace{TI_{EM(G)}(\mathbf{x}_o, \mathbf{y}_o)}_{\text{Graph Technical Inefficiency}} + \underbrace{AI_{GDA(G)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}}, \tilde{\mathbf{p}})}_{\text{Norm. Allocative Inefficiency}} \geq 0$$

Additive Models: Cost and Revenue Efficiency (GM)

- Normalized Cost and Revenue Inefficiency: Input and Output Orientation

$$\underbrace{\frac{\sum_{m=1}^M w_m x_{om} - C(\mathbf{y}_o, \mathbf{w})}{NF_{EM(I)}}}_{\text{Norm. Cost Inefficiency}} = \underbrace{TI_{EM(I)}(\mathbf{x}_o, \mathbf{y}_o)}_{\text{Input Technical Inefficiency}} + \underbrace{AI_{EM(I)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{w}})}_{\text{Norm. Allocative Inefficiency}} \geq 0,$$

$$\underbrace{\frac{R(\mathbf{x}_o, \mathbf{p}) - \sum_{n=1}^N p_n y_{on}}{NF_{EM(O)}}}_{\text{Norm. Revenue Inefficiency}} = \underbrace{TI_{EM(O)}(\mathbf{x}_o, \mathbf{y}_o)}_{\text{Output Technical Inefficiency}} + \underbrace{AI_{EM(O)}(\mathbf{x}_o, \mathbf{y}_o, \tilde{\mathbf{p}})}_{\text{Norm. Allocative Inefficiency}} \geq 0.$$

Multiplicative Models: Profitability Efficiency (1/2)

- Normalized Profitability Inefficiency: Technological axioms (VRS & CRS)

- Observed profitability: $\Gamma_o = p \cdot y_o / w \cdot x_o$

- Maximum profitability: $\Gamma(w, p) = \max_{x, y} \left\{ p \cdot y / w \cdot x \mid x \geq X\lambda, y \leq Y\lambda, \lambda \geq 0 \right\}$

- Technical efficiency: $TE_{GDF}^{CRS}(x, y, \alpha) \leq 1 \Rightarrow$ Duality

- Fenchel-Mahler Inequality:

$$\Gamma E(x_o, y_o, w, p) = \frac{\Gamma_o}{\Gamma(p, w)} = \frac{p \cdot y_o / w \cdot x_o}{\Gamma(p, w)} = \frac{\sum_{n=1}^N p_n y_{on} / \sum_{m=1}^M w_m x_{om}}{\Gamma(p, w)} \leq TE_{GDF}^{CRS}(x, y, \alpha)$$

Multiplicative Models: Profitability Efficiency (2/2)

Technical Efficiency: TE

$$\begin{aligned} TE_{GDF}^{\text{CRS}}(\mathbf{x}_o, \mathbf{y}_o, \alpha) &= \min_{\delta, \boldsymbol{\lambda}} \delta^{\text{CRS}} \\ \text{s.t. } & \sum_{j=1}^J \lambda_j x_{jm} \leq (\delta^{\text{CRS}})^{1-\alpha} x_{om}, \quad m = 1, \dots, M, \\ & \sum_{j=1}^J \lambda_j y_{jn} \geq y_{on}/(\delta^{\text{CRS}})^\alpha, \quad n = 1, \dots, N, \\ & \lambda_j \geq 0. \end{aligned}$$

Profitability Efficiency: PE

$$\frac{\sum_{n=1}^N p_n y_{on} / \sum_{m=1}^M w_m x_{om}}{\Gamma(\mathbf{p}, \mathbf{w})} = \underbrace{\frac{TE_{GDF}^{\text{CRS}}(\mathbf{x}_o, \mathbf{y}_o, \alpha)}{\text{Graph Technical Efficiency CRS}}}_{\text{Profitability Inefficiency}} \times \underbrace{\frac{AE_{GDF}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p})}{\text{Allocative Inefficiency}}}_{\text{Allocative Inefficiency}}$$
$$\underbrace{\frac{TE_{GDF}(\mathbf{x}_o, \mathbf{y}_o, \alpha)}{\text{Graph Technical Efficiency VRS}}}_{\text{Graph Technical Efficiency VRS}} \times \underbrace{\frac{SE_{GDF}(\mathbf{x}_o, \mathbf{y}_o, \alpha)}{\text{Scale Efficiency}}}_{\text{Scale Efficiency}} \times \underbrace{\frac{AE_{GDF}(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}, \mathbf{p})}{\text{Allocative Inefficiency}}}_{\text{Allocative Inefficiency}} \geq 0.$$

What Economic Model Is Better? (1/2)

- 1) General Considerations: Aim of study, Input or Output Discretionarity, Data Availability \Rightarrow Duality drives the choice of technical (in)efficiency
- 2) Properties:
 - Technical (in)efficiency:
 - ✓ Indication, Commensurability, Monotonicity, Homogeneity, Translation inv.,...
 - Economic (in)efficiency:
 - ✓ Commensurability, Indication, Price Homogeneity, Quantity Homogeneity,...
 - Allocative (in)efficiency: (Session 1A CEYP).
 - ✓ Essentiality, Extended Essentiality.
- 3) There is a trade-off between Indication (TE) and Essentiality (AE)

What Economic Model is better? (2/2)

| Economic model | Multiplicative models | | Additive models | | | | | | | | |
|--|---|--|--|-----------------------------------|--|-----------------------------|------------------------|----------------------------|---------------------------|---------------------------------|--|
| | Oriented models: Cost and revenue eff. | Graph model: Profitability efficiency | Cost, revenue, and profit inefficiency | | | | | | | | |
| Measure | Farrell/ Shephard DFs (Chap. 3) | Generalized DF (Chap. 4) | Russell IM (Chap. 5) | Weighted Additive DF (Chap. 6) | Enhanced Russell Graph IM (Chap. 7) | Directional DF (Chap. 8) | Hölder DF (Chap. 9) | Modified DDF (Chap. 11) | Reverse DDF (Chap. 12) | General Direct IM (Chap. 13) | |
| Technical Eff. Properties (Sect. 2.2.4) | E1. Indication ⁽¹⁾ | ✗ | ✗ | ✓ | ✓ | ✓ | ✗ | ✗ | ✓ | ✓ | |
| | E2. Monotonicity ⁽²⁾ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | |
| | E3. Homogeneity | ✓ | ✓ ⁽³⁾ | ✗ | ✗ | ✗ | ✗ ⁽⁴⁾ | ✗ | ✗ | ✗ | |
| | E4. Commensurab. ⁽⁵⁾ | ✓ | ✓ | ✓ | ✓ ⁽⁶⁾ | ✗ | ✗ ⁽⁷⁾ | ✗ ⁽⁸⁾ | ✗ ⁽⁹⁾ | ✗ ⁽¹⁰⁾ | |
| | E5. Traslation Inv. ⁽¹¹⁾ | ✓ ⁽¹²⁾ | ✗ | ✓ | ✓ | ✗ | ✓ ⁽¹³⁾ | ✓ | ✓ | ✗ | |
| Economic Eff. Properties (Sect. 2.3.5) | P1. Boundness | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | |
| | P2. Indication | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | |
| | P3. Price Homog. | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | |
| | P4. Quant. Homog. | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | |
| | P5. Commensurab. ⁽¹⁴⁾ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✓ | ✗ | |
| Allocative Eff. Properties (Sect. 2.4.5) | D1. Essentiality | ✓ | ✓ | ✗ | ✗ | ✗ | ✓ | ✓ | ✗ | ✓ | |
| | D2. Refined Essent. | ✓ | ✓ | ✗ | ✗ | ✗ | ✓ ⁽¹⁵⁾ | ✗ | ✗ ⁽¹⁵⁾ | ✓ ⁽¹⁵⁾ | |

Notes: ⁽¹⁾ Pareto -Koopmans Efficiency; ⁽²⁾ Weak monotonicity; ⁽³⁾ Satisfies almost homogeneity; ⁽⁴⁾ Satisfies translation property; ⁽⁵⁾ Units invariance; ⁽⁶⁾ Yes for weights different from one; ⁽⁷⁾ Yes for the proportional DDF with $(g_x, g_y) = (x_o, y_o)$; ⁽⁸⁾ Yes for weighted norms; ⁽⁹⁾ Yes for the proportional MDDF with $(g_x, g_y) = (x_o, y_o)$; ⁽¹¹⁾ For variable returns to scale; ⁽¹²⁾ For the input orientation it is translation invariant for output, and vice versa; ⁽¹³⁾ Yes, under specific condition for the directional vector, Aparicio et al. (2015); ⁽¹⁴⁾ Units invariance; ⁽¹⁵⁾ Yes when the normalization factor is the same all firms: $p \cdot g_y + w \cdot g_x$.

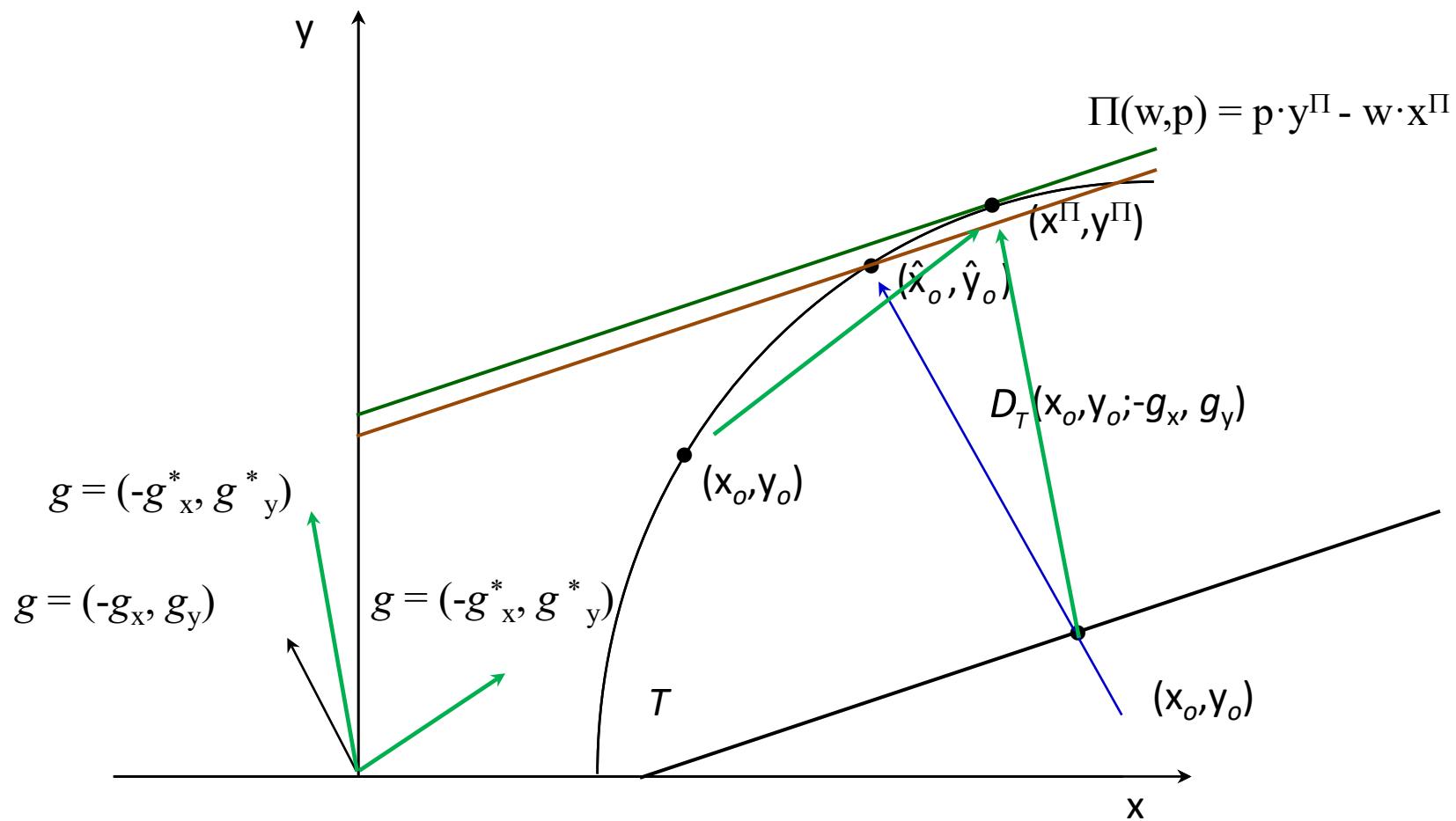
Extensions

Endogenous models (1/2)

- The decomposition of economic efficiency depends on the exogenous and subjective choice of model; e.g. above directional vector (DDF).
- Zofío, Pastor and Aparicio (2013) endogenize the directional vector (DDF) in the profit decomposition \Rightarrow This provides inconsistent strategies when solving economic inefficiencies that result in adjustments costs.
- Observations are directly projected on the economic benchmark (e.g. maximum profit, minimum costs) \Rightarrow Profit (in)efficiency is then technical or allocative efficient (deemed “too extreme” by Petersen (2018)).
- This model/idea has been recently applied by Kapelko, Oude-Lansink and Zofío (2022) and Pastor, Zofío et al. (2022) (Session 1A CEYP).

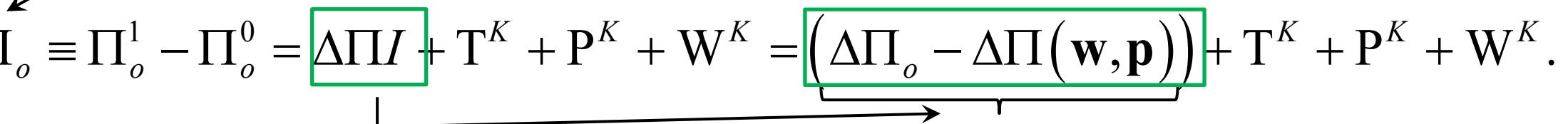
Endogenous models (2/2)

- Profit and profitability efficiency decomposition



Profit Change (Cost, Revenue, Profitability) (1/2)

- Economic efficiency is a building block in the decomposition of economic efficiency *change* (Profit, Cost, Revenue, Profitability). ([GT and Lovell, 2015](#))
- [Aparicio and Zofío \(2022\)](#) introduce exact decomposition of profit change using Konüs and Bennet indicators of changes in prices and quantities ([Diewert, 2014, Balk and Zofío, 2019](#)).
- Profit change: $\Delta\Pi_o \equiv \Pi_o^1 - \Pi_o^0 = \mathbf{p}^1 \cdot \mathbf{y}_o^1 - \mathbf{w}^1 \cdot \mathbf{x}_o^1 - (\mathbf{p}^0 \cdot \mathbf{y}_o^0 - \mathbf{w}^0 \cdot \mathbf{x}_o^0)$
- Then, profit inefficiency can be decomposed into profit inefficiency change, technological change, and output and input price changes:

$$\Delta\Pi_o \equiv \Pi_o^1 - \Pi_o^0 = \boxed{\Delta\Pi I} + T^K + P^K + W^K = \boxed{(\Delta\Pi_o - \Delta\Pi(w, p))} + T^K + P^K + W^K.$$


Profit Change (Cost, Revenue, Profitability) (2/2)

- where:

$$\Delta \Pi I_o = \Pi I_o^0 - \Pi I_o^1 = \left(\Pi^0(w^0, p^0) - \Pi_o^0 \right) - \left(\Pi^1(w^1, p^1) - \Pi_o^1 \right)$$

- We can now decompose profit inefficiency resorting to any of the above models. Relying on the DDF we obtain:

$$\Delta \Pi I_o = \Delta \Pi_o - \Delta \Pi(w, p) = \Delta TI_{DDF} + \Delta AI_{DDF}$$

- And substituting into the profit change expression we obtain:

$$\Delta \Pi_o \equiv \Pi_o^1 - \Pi_o^0 = \Delta \Pi I + T^K + P^K + \Omega^K = \boxed{\Delta TI_{DDF} + \Delta AI_{DDF}} + T^K + P^K + \Omega^K.$$

Conclusions

Conclusions

- Economic efficiency (profit, profitability, cost, or revenue) **can be consistently decomposed resorting to duality theory.**
- Any (in)efficiency measure (distance function) that is capable of characterizing the production technology (representation property) **can be used to decompose economic efficiency** into technical and allocative components. All decompositions are different!!!
- At the moment, **no technical (in)efficiency measure satisfies all desired properties:** Indication vs. Essentiality. Current work on this issue.
- The decomposition of economic efficiency **is subjective as it depends on the choice of model** (e.g., slacks, Hölder, DDF,...) \Rightarrow Endogenous models.
- Profit efficiency enters the **decomposition of profit change** (profit., cost, rev.)